Math and Sudoku

Exploring Sudoku boards through graph theory, group theory, and combinatorics

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Math 401
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Winter 2016

Abstract

Encoding Sudoku puzzles as partially colored graphs, we state and prove Akman's theorem [1] regarding the associated partial chromatic polynomial [2]; we count the 4x4 sudoku boards, in total and fundamentally distinct; we count the diagonally distinct 4x4 sudoku boards; and we classify and enumerate the different structure types of 4x4 boards.

Introduction

Sudoku is a logic-based puzzle game relating to Latin squares [3]. In the most common size, 9x9, each row, column, and marked 3x3 block must contain the numbers 1 through 9. A sudoku board can be formed for any n ∈ ℕ, with the resulting board having n n x n blocks and total size n^2x n^2. In this investigation, we will use n = 2 for our boards, for 4x4 sudoku. We selected this size, as opposed to the standard 9x9, for ease of calculation: there exist somewhere on the order 10^{21} 9x9 boards [4]. As will be shown, there exist a great deal fewer 4x4 boards.

I. Sudoku Puzzles and Boards as Graphs, and Partial Chromatic Polynomials

Considering a sudoku board as a mathematical object, it is useful to encode a completed board as a graph: each vertex corresponds to a cell in the board, and two distinct vertices are adjacent iff the two cells share a row, column, or n x n block.

Let a sudoku *board* be a completed sudoku puzzle, so that each of the n^4 cells contains a digit. Let a sudoku *puzzle* be a **partially completed** sudoku board: that is, at most $n^4 - 1$ cells contain digits. Any cell containing a digit can thus be encoded as a colored vertex. If a board is properly solved—i.e., the rules of sudoku are respected and every cell contains a digit—then the graph has a proper coloring. Any puzzle, then, corresponds to a *partial coloring* of at least one board. A *well-formed puzzle* corresponds to exactly one board: a well-formed puzzle has a unique solution. This is not the case for every puzzle; indeed, a puzzle that is not well-formed may not have a solution, and will not correspond to any board.

Note: individual cells shall be referred to by their $(row, column)$ coordinates. An equivalence class of a cell shall comprise all cells currently marked with, or to include, the same entry as the cell.

Take, for example, the puzzle:

	2		4
3			2
2	3	4	1
4	1	2	3

Figure 1: A 4x4 sudoku puzzle.

As a partially colored graph, we have equivalence classes:
"1" $= [(1,1)] = \{(1,1),(3,4)\}$; "2" $= [(1,2)] = \{(1,2),(2,4),(3,1),(4,3)\}$;
"3" $= [(2,1)] = \{(2,1),(3,2),(4,4)\}$; "4" $= [(1,4)] = \{(1,4),(3,3),(4,1)\}$.
Any board correspond to this puzzle will be a completion of this puzzle. Any properly colored board corresponding to the coloring of this puzzle will be *consistent* with the partial coloring of the puzzle.

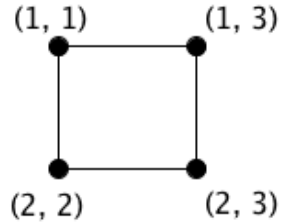
According to the standard rules of sudoku, only n^2 colors, or the numbers $\{1, \ldots, n\}$ may be used to complete the coloring/board. However, the only requirement for a proper coloring to be consistent with the partial coloring is that the already-colored vertices retain their colors (this also implies that the existing equivalence classes and independent sets retain their current members). Then the minimum number of colors required for a proper, consistent, completion of any partial coloring is equal to the number of colors present in the partial coloring, which is the same as the number of distinct digits appearing in the puzzle. The maximum number of colors which may appear in a proper, consistent, completion of a partial coloring is equal to the number of blank cells plus the number of colors appearing in the puzzle: to be consistent, a proper completion must not change any of the colors used, and so will require at least that many colors. Going beyond standard sudoku play, each blank cell may receive an unused distinct digit. Then the greatest number of colors for any proper, consistent completion is the sum of the number of distinct clues and the number of empty cells.

Any two vertices with the same color share an equivalence class; further they are in the same independent set. Note that there is a one-to-one correspondence between independent sets and color classes. Any un-colored vertex (empty cell) may be placed into an existing independent set, provided that the coloring remains proper (the rules of sudoku are respected), or it may be placed into a new independent set. Call a proper completion of the partial coloring *generic* if it is merely a partitioning of the vertices into independent sets: that is, no colors assigned to the empty cells while they are placed into color classes.

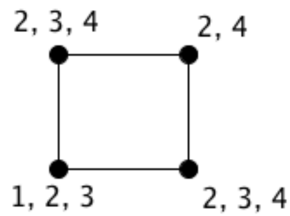
Using the technique of deletion-contraction, the chromatic polynomial of a graph can be found: $\chi(G,k)$, where $\chi(G,k)$ equals the number of proper vertex colorings of the graph G using at most k colors. This technique can also be used on a partially colored graph, to generate the partial chromatic polynomial. If we observe the same puzzle as *Fig. 1*, we see that cell $(1,3)$ must avoid two colors, those of color classes "2" and "4". More specifically, the cell $(1,3)$ cannot share a color class with equivalence classes $[(1,2)]$ & $[(1,4)]$.

	2		4
3			2
2	3	4	1
4	1	2	3

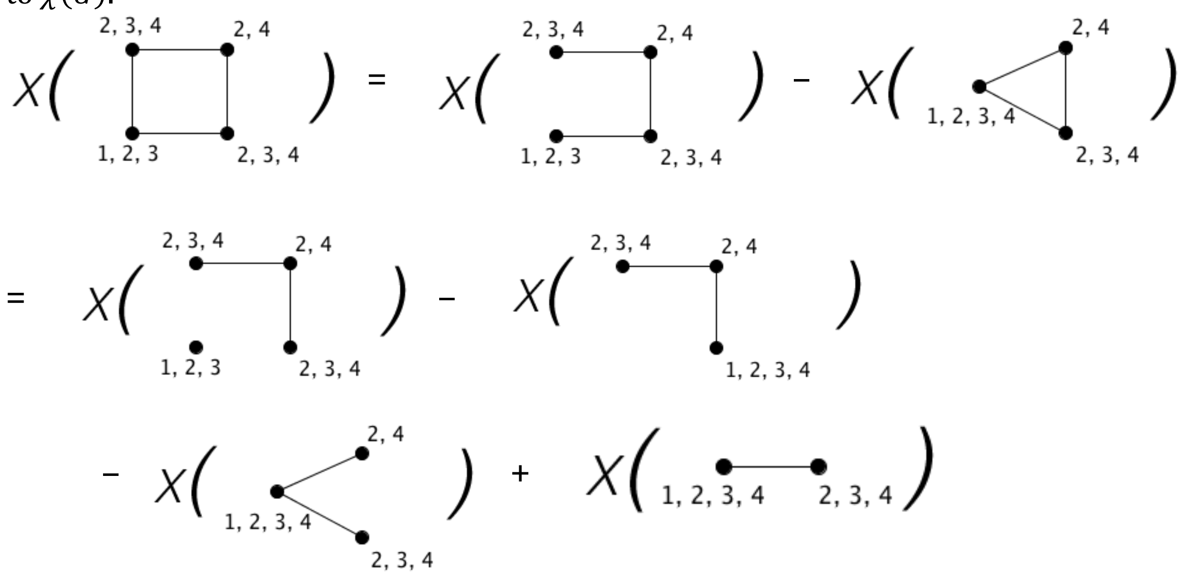
We shall encode the empty cells as a graph (with the vertices labeled with their grid coordinates):

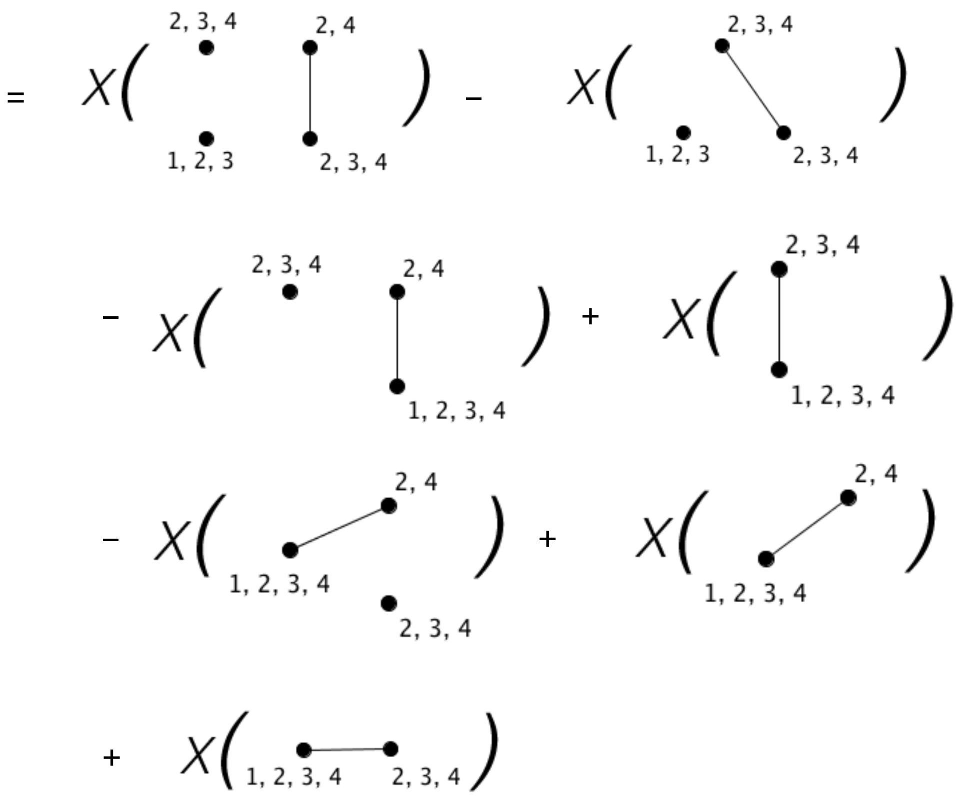


We shall now encode it as a partially colored graph, replacing the vertex labels with the color classes each vertex must avoid for a proper coloring:



When applying deletion-contraction, any vertex formed by contracting an edge shares the adjacencies, and thus the coloring restrictions, of the previously distinct vertices. Then the restricted color classes of a contracted-edge vertex correspond to the union of the restrictions of the distinct vertices. We now apply deletion-contraction to our sample partial coloring, using the recursion formula for the chromatic polynomial, $\chi(G,k) = \chi(G-e,k) - \chi(G\cdot e,k)$. Note: we shorten $\chi(G,k)$ to $\chi(G)$.





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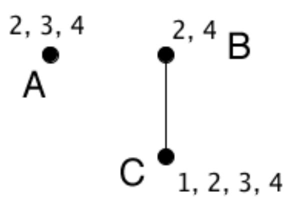

, we see that vertex C must avoid four colors; then for a k-coloring, it can receive any of $k-4$ colors. Vertex B must avoid colors "2" and "4", as well as whichever color is placed on vertex C; then B can receive any of $k-3$ colors. Vertex A must avoid colors "2", "3", and "4", and so can receive any of $k-3$ colors. Then the term for this portion of the chromatic polynomial is $(k-4)(k-3)^2$. Applying this reasoning to every term, we find that chromatic polynomial with restrictions of our puzzle is:

$$\chi(G,k) = (k-3)^4 - (k-3)^2(k-4) - (k-4)(k-3)^2 + (k-4)^2 - (k-4)(k-3)^2 + (k-4)(k-3) + (k-4)^2$$

$$\chi(G,k) = k^4 - 15k^3 + 87k^2 - 230k + 233$$

In order for any coloring to be consistent with the puzzle, k must be at least as large as the number of distinct colors already used: here, $k \geq 4$.

We see that $\chi(4) = 1$; indeed, this puzzle has one possible completion in line with standard sudoku rules:

	2		4
3			2
2	3	4	1
4	1	2	3

$\rightarrow$

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

We also see that $\chi(5) = 8$. This indicates that if 5 colors, or the digits $\{1, \ldots, 5\}$ were to be allowed, then there are 8 possible consistent boards:

5	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

5	2	1	4
3	4	5	2
2	3	4	1
4	1	2	3

5	2	3	4
3	4	5	2
2	3	4	1
4	1	2	3

1	2	3	4
3	5	1	2
2	3	4	1
4	1	2	3

1	2	5	4
3	5	1	2
2	3	4	1
4	1	2	3

1	2	3	4
3	4	5	2
2	3	4	1
4	1	2	3

1	2	5	4
3	4	1	2
2	3	4	1
4	1	2	3

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

Note that a proper k-coloring is a proper coloring using *at most k* colors: allowing 5 colors does not prevent a 4-coloring. Then $\chi(5)$, the number of completions using at most 5 digits, is equal to $\chi(4)$, the number of completions using at most 4 digits (here, there is one), plus the number of completions using exactly 5 digits (here, there are seven).

Let n now be the number of vertices in the graph encoding of the sudoku grid (for 9x9, $n = 81$; for 4x4, $n = 16$); let t be the number vertices already colored (or the number of clues appearing in the puzzle); let λ_0 be the number of distinct colors in the partial coloring (or the number of distinct digits among the clues). In our example,

	2		4
3			2
2	3	4	1
4	1	2	3

, $n = 16; t = 12;\ \lambda_0 = 4$.

We now state Akman's theorem [1]: "Let G be a graph with n vertices, and C be a partial proper coloring of t vertices of G using exactly λ_0 colors. Define $p_{G,C}(\lambda)$ to be the number of ways C can be completed to a proper λ-coloring of G. Then for $\lambda \geq \lambda_0$, the expression $p_{G,C}(\lambda)$ is a monic polynomial of degree $n - t$."

Proof: Following our earlier discussion, the partial proper coloring, C, of t vertices, is the puzzle with t clues, λ_0 of which are distinct digits. For $\lambda = \lambda_0$, no new colors may be added (no new digits may be used). That is, there shall remain exactly λ_0 color classes in the completed board; since these colors are already assigned, there is one way color assignment. If the board has $\lambda = \lambda_0 + 1$ independent sets, then since the puzzle already has λ_0 independent sets, all cells belonging to any existing color class must retain the already- assigned color; any cells belonging to the new color class will receive the $\lambda - \lambda_0 = 1$ new color.

As previously discussed, the maximum number of color classes appearing in any board is the sum of the number of already appearing color classes and the number of empty cells. Then the total number of color classes/independent sets is here $\lambda_0 \leq \lambda \leq (n - t) + \lambda_0$. If we let $r = \lambda - \lambda_0$, the number of new color classes, then $0 \leq r \leq n - t$.

We apply induction to $\lambda > \lambda_0$: for each additional new color class, all previously assigned colors must be avoided, and we find a falling factorial of possible color choices for each of the additional r color classes, up to $r = n - t$. Since $r = 0$ is included (and applies to no new colors), then there are $(\lambda - \lambda_0)(\lambda - \lambda_0 - 1) \cdots (\lambda - \lambda_0 - r + 1)$ ways of applying colors to the color classes for the board partition wherein each blank cell is in its own color class.

Then we have determined the number of ways of assigning colors to the $\lambda_0 + r$ color classes of a given partition of the board. Call the number of ways of partitioning the $n-t$ empty cells into exactly r color classes $m_r(G,C)$. For each distinct partitioning of the puzzle and empty cells into color classes, there are $(\lambda-\lambda_0)(\lambda-\lambda_0-1)\cdots(\lambda-\lambda_0-r+1)$ ways of assigning colors. Then where $\chi(k)$, the chromatic polynomial evaluated at k, indicates the number of complete, consistent colorings of a λ_0-partial-coloring using *at most* k colors, $\chi(k)$ equals the sum of the number of completions using exactly λ_0 colors, exactly λ_0+1 colors,..., exactly k colors. Then $\chi(k)=\sum_{r=0}^{n-t} m_r(G,C)(\lambda-\lambda_0)(\lambda-\lambda_0-1)\cdots(\lambda-\lambda_0-r+1)=p_{G,C}(\lambda)$. Each term is a polynomial of degree r; the last term is of degree $r=n-t$, and correspoinds to the complete colorings using exactly $n-t$ new colors, or partitioning the empty cells into exactly $n-t$ new color classes. Since there are exactly $n-t$ empty cells/un-colored vertices, then there is only one such partition. And since each term of degree r, for $0\le r\le n-t$, the sum of the terms, $p_{G,C}$, is monic of degree $n-t$, and corresponds directly to the chromatic polynomial of the sudoku graph with initial coloring restraints on the empty cells. ◊

II. Enumeration of 4x4 Sudoku Boards

A. Total Enumeration:

Claim: ∃ 288 total 4x4 sudoku boards.

Proof: Pick an arbitrary filling-in of the upper left 2x2 block:

	1	2	3	4
1	a	b		
2	c	d		
3				
4				

Turning toward the lower right block, we recognize that “d” must appear in one of the 4 spots {(3, 3), (3, 4), (4, 3), (4, 4)}. We condition four cases:

I	1	2	3	4
1	a	b		
2	c	d		
3			d	
4				

II	1	2	3	4
1	a	b		
2	c	d		
3				d
4				

III	1	2	3	4
1	a	b		
2	c	d		
3				
4			d	

IV	1	2	3	4
1	a	b		
2	c	d		
3				
4				d

For each case, we subscript the possible entries for all remaining empty cells:

I	1	2	3	4
1	a	b	c	d
2	c	d	ab	ab
3	b	ac	d	ac
4	d	ac	ab	abc

II	1	2	3	4
1	a	b	d	c
2	c	d	ab	ab
3	b	ac	ac	d
4	d	ac	abc	ab

III	1	2	3	4
1	a	b	c	d
2	c	d	ab	ab
3	d	ac	ab	abc
4	b	ac	d	ac

IV	1	2	3	4
1	a	b	d	c
2	c	d	ab	ab
3	d	ac	abc	ab
4	b	ac	ac	d

We note that for each case, there is a single cell with three possible entries: {I: (4, 4), II: (4, 3), III: (3, 4), IV: (3, 3)}. Any cell with a single option is filled, and we condition subcases:

Ia	1	2	3	4
1	a	b	c	d
2	c	d	ab	ab
3	b	ac	d	ac
4	d	ac	ab	a

Ib	1	2	3	4
1	a	b	c	d
2	c	d	ab	ab
3	b	ac	d	ac
4	d	ac	ab	b

Ic	1	2	3	4
1	a	b	c	d
2	c	d	ab	ab
3	b	ac	d	ac
4	d	ac	ab	c

IIa	1	2	3	4
1	a	b	d	c
2	c	d	ab	ab
3	b	ac	ac	d
4	d	ac	a	ab

IIb	1	2	3	4
1	a	b	d	c
2	c	d	ab	ab
3	b	ac	ac	d
4	d	ac	b	ab

IIc	1	2	3	4
1	a	b	d	c
2	c	d	ab	ab
3	b	ac	ac	d
4	d	ac	c	ab

IIIa	1	2	3	4
1	a	b	c	d
2	c	d	ab	ab
3	d	ac	ab	a
4	b	ac	d	ac

IIIb	1	2	3	4
1	a	b	c	d
2	c	d	ab	ab
3	d	ac	ab	b
4	b	ac	d	ac

IIIc	1	2	3	4
1	a	b	c	d
2	c	d	ab	ab
3	d	ac	ab	c
4	b	ac	d	ac

IVa	1	2	3	4
1	a	b	d	c
2	c	d	ab	ab
3	d	ac	a	ab
4	b	ac	ac	d

IVb	1	2	3	4
1	a	b	d	c
2	c	d	ab	ab
3	d	ac	b	ab
4	b	ac	ac	d

IVc	1	2	3	4
1	a	b	d	c
2	c	d	ab	ab
3	d	ac	c	ab
4	b	ac	d	ac

Recognizing that these cases limit the options for the remaining cells, we fill out according to the rules:

Ia	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	a	d	c
4	d	c	b	a

Ib	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	b	a	d	c
4	d	c	a	b

Ic	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	c	d	a
4	d	a	b	c

IIa	1	2	3	4
1	a	b	d	c
2	c	d	b	a
3	b	a	c	d
4	d	c	a	b

IIb	1	2	3	4
1	a	b	d	c
2	c	d	a	b
3	b	a	c	d
4	d	c	b	a

IIc	1	2	3	4
1	a	b	d	c
2	c	d	b	a
3	b	c	a	d
4	d	a	c	b

IIIa	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	d	c	b	a
4	b	a	d	c

IIIb	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	d	c	a	b
4	b	a	d	c

IIIc	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	d	a	b	c
4	b	c	d	a

IVa	1	2	3	4
1	a	b	d	c
2	c	d	b	a
3	d	c	a	b
4	b	a	c	d

IVb	1	2	3	4
1	a	b	d	c
2	c	d	a	b
3	d	c	b	a
4	b	a	c	d

IVc	1	2	3	4
1	a	b	d	c
2	c	d	b	a
3	d	a	c	b
4	b	c	a	d

We see that each of the 4 cases results in 3 singular outcomes, based on an arbitrary ordering of the upper left block. As (a, b, c, d) can represent any permutation of the numbers 1 through 4, there are $4! = 24$ possible orderings of the upper left block, each with 12 distinct outcomes. $\therefore \exists\ 4! \cdot 12 = 24 \cdot 12 = 288$ completed 4x4 sudoku boards. ◊

It is interesting to note that of the 576 order 4 Latin squares [3], exactly half of them are 4x4 sudoku boards.

B. Distinct Enumeration:

Claim: ∃ 2 fundamentally different 4x4 sudoku boards.

Proof: Let S be the set of all 288 4x4 sudoku boards. Any $x, y \in S$ will be considered equivalent if any group action or combination of group actions on x yield(s) y. Let $G = < B, C, r_1, r_3, c_1, c_3, D_8 > \cup S_4$, the group generated by the listed actions and the group of permutations on $\{1, \dots, 4\}$, the listed actions defined thusly:

–B places the first two rows (in order) in the position of the last two rows, and the last two rows (in order) in the position of the first two rows;
–C places the first two columns (in order) in the position of the last two columns, and the last two columns (in order) in the position of the first two columns;
–r_1 swaps the row positions of rows 1 and 2;
–r_3 swaps the row positions of rows 3 and 4;
–c_1 swaps the column positions of rows 1 and 2;
–c_3 swaps the column positions of columns 3 and 4;
–D_8 indicates the symmetries of the square: $\{I, R, R^2, R^3, F, FR, FR^2, FR^3\}$

It can be observed that, given a legal sudoku board, the legality is preserved by each of these actions: for the symmetries of the square, all rows, columns, and 2x2 blocks are preserved, while columns may become rows and vice versa, and the order of all rows and/or columns may be reversed; for the first six actions, any 2x2 block is preserved, and if the permutation of a row or column is altered, so too are the permutations of all intersecting rows/columns. Further, given that function composition is associative, any combination of these actions also preserves legality.

We consider any two sudoku boards equivalent iff any composition of these actions and/or S_4 transforms one to the other. Of the 288 total boards, we found that a board with any given upper left block was permutable to 24 others. Then there are at most 288 ÷ 24 = 12 non-equivalent boards. We examine the 12 subcases found, and find that some are indeed transformations of others:

Ia

	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	a	d	c
4	d	c	b	a

$\xrightarrow{c_3}$

	1	2	3	4
1	a	b	d	c
2	c	d	b	a
3	b	a	c	d
4	d	c	a	b

= **IIa**

Ia

	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	a	d	c
4	d	c	b	a

$\xrightarrow{r_3}$

	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	d	c	b	a
4	b	a	d	c

=**IIIa**

IIIa

	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	d	c	b	a
4	b	a	d	c

$\xrightarrow{c_3}$

	1	2	3	4
1	a	b	d	c
2	c	d	b	a
3	d	c	a	b
4	b	a	c	d

= **IVa**

Ib

	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	b	a	d	c
4	d	c	a	b

$\xrightarrow{c_3}$

	1	2	3	4
1	a	b	d	c
2	c	d	a	b
3	b	a	c	d
4	d	c	b	a

= **IIb**

Ib	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	b	a	d	c
4	d	c	a	b

$\xrightarrow{r_3}$

	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	d	c	a	b
4	b	a	d	c

= **IIIb**

IIIb	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	d	c	a	b
4	b	a	d	c

$\xrightarrow{c_3}$

	1	2	3	4
1	a	b	d	c
2	c	d	a	b
3	d	c	b	a
4	b	a	c	d

= **IVb**

Ic	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	c	d	a
4	d	a	b	c

$\xrightarrow{c_3}$

	1	2	3	4
1	a	b	d	c
2	c	d	b	a
3	b	c	a	d
4	d	a	c	b

= **IIc**

Ic	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	c	d	a
4	d	a	b	c

$\xrightarrow{r_3}$

	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	d	a	b	c
4	b	c	d	a

= **IIIc**

IIIc	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	d	a	b	c
4	b	c	d	a

$\xrightarrow{c_3}$

	1	2	3	4
1	a	b	d	c
2	c	d	b	a
3	d	a	c	b
4	b	c	a	d

= **IVc**

We have thus found that there are at most three equivalence classes of boards. By representative, they are:

A	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	a	d	c
4	d	c	b	a

B	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	b	a	d	c
4	d	c	a	b

C	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	c	d	a
4	d	a	b	c

We note that our transformations are actions of a group, with the operation of function composition; then combinations of transformations are themselves valid transformations:

B	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	b	a	d	c
4	d	c	a	b

$\xrightarrow{FR^3}$

	1	2	3	4
1	a	c	b	d
2	b	d	a	c
3	c	b	d	a
4	d	a	c	b

$\xrightarrow{(bc)}$

	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	c	d	a
4	d	a	b	c

= **C**

Then we see that there are at most two equivalence classes of 4x4 boards:

A	1	2	3	4
1	a	b	c	d
2	c	d	a	b
3	b	a	d	c
4	d	c	b	a

B	1	2	3	4
1	a	b	c	d
2	c	d	b	a
3	b	a	d	c
4	d	c	a	b

Notice that the only differences between boards A and B are the cell pairs in row 2, columns 3 & 4, and row 4, columns 3 & 4. In order to transform from A to B (and since G is a group, if such a transformation exists then an inverse transformation exists from B to A), we need to make the following cell changes: $(2,3) \leftrightarrow (2,4); (4,3) \leftrightarrow (4,4)$. If this is a valid transformation, then it is a combination of the generating actions; further, if it is a valid action, then when applied to a legal board, legality is preserved under the action. However, observe that this action, performed on a legal board, renders the board illegal (observe columns 3 and 4):

	1	2	3	4
1	a	c	b	d
2	b	d	a	c
3	c	b	d	a
4	d	a	c	b

→

	1	2	3	4
1	a	c	b	d
2	b	d	c	a
3	c	b	d	a
4	d	a	b	c

Then this action is not a legal transformation, and thus not a combination of any of the generation actions, and thus not a transformation belonging to our group. Then there do not exist any transformations $[A] \leftrightarrow [B]$. $\therefore \exists$ 2 fundamentally different 4x4 sudoku boards. ◊

We also see that $|[A]| = 24 \cdot 4 = 96$, and $|[B]| = 24 \cdot 8 = 192$.

III. Diagonal 4x4 Sudoku Boards

A: Distinct Enumeration:

We turn our attention to diagonally distinct sudoku boards: those with the added constraint that each of the main diagonals must also contain the numbers {1, 2, 3, 4}. (See Akman [1] for a proof of existence of diagonally distinct boards for any size sudoku.)

Claim: There is only one distinct diagonal 4x4 sudoku board.

Proof: Given that there are two distinct boards overall, then there must be at most 2 distinct diagonal boards. We see that board IIIa, of [A], satisfies:

IIIa	1	2	3	4
1	**a**	b	c	**d**
2	c	**d**	**a**	b
3	d	**c**	**b**	a
4	**b**	a	d	**c**

However, [B] does not readily present such a board. Observe that diagonality is a structural feature, and does not change under permutation: For any diagonal where not all of {a, b, c, d} are present, no permutation {a, b, c, d} → S_4 can add any of the missing element(s) (as permutations are one-to-one). Equivalence class [B] was found by collapsing 8 subcases, all transformable to each other. The 8 subcases each represented 24 boards, all achievable by permutation. Then since none of the 8 subcases are themselves not diagonal, then none of the permutations represented by the 8 subcases are themselves diagonal. There are 288 boards total; 96 are in [A], leaving 192 in [B]. As observed, there are 8 boards structurally non-diagonal, each representing 24 non-diagonal permutations. Then [B] contains $8 \cdot 24 = 192$ non-diagonal boards ⇒ [B] contains no diagonal boards. Then since a single diagonal board has been found in [A], all other diagonal boards must be reachable by some combination of transformations and permutations. ∴ ∃ 1 fundamentally distinct 4x4 sudoku board. ◊

B. Total Enumeration:

Claim: ∃ 48 total diagonal 4x4 sudoku boards.

Proof: Observe that boards IIa and IIIa, of equivalence class [A], are both diagonal, whereas boards Ia and IVa are not. Each of IIa and IIIa represent 24 boards by permutation; then there are at least $24 \cdot 2 = 48$ diagonal boards.

In order to prove that there are at most 48 diagonal boards, we turn our attention to the different structures of the 4x4 boards, and which group actions preserve or break those structures.

C. Structures of 4x4 Board Structures and Group Actions:

It is obvious that all of S_4 as well as the actions of D_8 preserve diagonality of a board. We focus on $\{B, C, r_1, r_3, c_1, c_3\}$. Observe that under R, a 90º clockwise rotation, all rows become columns and all columns become rows. Then under $R, B \equiv C, r_1 \equiv c_3, r_3 \equiv c_1$. Then we may focus only on $\{B, C, r_1, r_3\}$. Consider the diagonal board IIIa:

IIIa	1	2	3	4
1	**a**	b	c	**d**
2	c	**d**	**a**	b
3	d	**c**	**b**	a
4	**b**	a	d	**c**

Under r_1, this is transformed thusly:

IIIa

a	b	c	d
c	d	a	b
d	c	b	a
b	a	d	c

$\xrightarrow{r_1}$

c	d	a	**b**
a	**b**	**c**	d
d	**c**	**b**	a
b	a	d	**c**

Clearly, diagonality is not preserved. And under r_3:

IIIa

a	b	c	d
c	d	a	b
d	c	b	a
b	a	d	c

$\xrightarrow{r_3}$

a	b	c	**d**
c	**d**	**a**	b
b	**a**	**d**	c
d	c	b	**a**

Again, diagonality is not preserved.

However, diagonality is preserved under B (and thus under C): For the two cells (4, 1), (3, 2), in diagonal board these are two of the four distinct digits. The other two of the four must be present, in some order, in cells (1, 4) and (2, 3). The same is true of the other two blocks, for the upper left- lower right diagonal. Then the digits in the four cells (3, 1), (4, 2), (1, 3), (2, 4), must also be the four distinct digits. Under the action B, the cells (3, 1) and (4, 2) become the top two digits of the upper- left lower right diagonal, while cells (1, 3) and (2, 4) become the lower two digits of the diagonal. Similarly for the upper right- lower left diagonal. Then B preserves diagonality; by R or R^3, so too does C.

Consider the two non-diagonal representatives of equivalence class [A]:

Ia

a	b	c	d
c	d	a	b
b	a	d	c
d	c	b	a

IVa

a	b	d	c
c	d	b	a
d	c	a	b
b	a	c	d

Just as $\{D_8, S_4, B, C\}$ preserve diagonality, the same argument holds for these actions preserving non-diagonality. Indeed, the actions $\{r_1, r_3, c_1, c_3\}$ are the only actions which will transform diagonal boards to non-diagonal boards and vice versa.

Consider the diagonal entries of the two boards above. Board Ia has as entries on its main diagonals {a, d}; board Iva has entries {ad}, {bc}. We will classify board structures based on their diagonal entries. Board Ia and its 24 permutations have a total of 2 distinct digits along the main diagonals. Board IVa and its 24 permutations have 2 distinct digits along each of its main diagonals. Boards IIa and IIIa, being the diagonal boards, of course have all of the distinct digits along each of their main diagonals. We see, then that equivalence class [A] can be further structured into three equivalence subclasses: [Ia], which contains 24 boards, each of which have a total of 2 diagonal digits; [IIa] = [IIIa], which contains 48 boards, each of which are diagonal; and [IVa], which contains 24 boards, each of which has 2 distinct digits along each of its diagonals.

These account for all 96 of the boards of [A]; as there are no diagonal boards in [B], then there are at most 48 diagonal boards; given the lower bound found earlier, then there are exactly 48 diagonal 4x4 boards. ◊

We have seen board structures, all of equivalence class [A], which contain either 2 or 4 distinct digits along the diagonals. What of the boards of [B]? Indeed, all of the boards of [B] each contain 3 distinct digits along each diagonal.

It can be seen (at the end of section II.A) that each of the 8 representatives of equivalence class [B] = [C] have three distinct digits along each main diagonal. It is clear that no permutation of S_4 will change the number of distinct digits along a diagonal. Here, we show that these 8 boards (and their permutations) are all of the possible 4x4 sudoku boards with 3 distinct digits across the diagonals.

We condition boards based on the upper left- lower right diagonal. Given a sequence _ _ | _ _, we see that there are four structural possibilities: the first cell matches the third, the first cell matches the fourth, the second cell matches the third, the second cell matches the fourth; with each of the two remaining cells distinct. Without loss of generality, we note the possibilities:

1)

a			
	b		
		a	
			c

2)

a			
	b		
		c	
			a

3)

a			
	b		
		b	
			c

4)

a			
	b		
		c	
			b

Recognizing that these cases limit the options for the remaining cells, we fill out according to the rules:

1)

a	cd	bcd	bd
cd	b	cd	a
bcd	cd	a	bd
bd	a	bd	c

2)

a	cd	bd	bcd
cd	b	a	cd
bd	a	c	bd
bcd	cd	bd	a

3)

a	cd	cd	b
cd	b	acd	ad
acd	acd	b	ad
b	ad	ad	c

4)

a	cd	b	cd
cd	b	ad	acd
b	ad	c	ad
cd	acd	ad	b

We pick an arbitrary single cell with two options, condition, and fill out:

1a)

a	**c**	b	d
d	b	c	a
c	d	a	b
b	a	d	c

1b)

a	**d**	c	b
c	b	d	a
b	c	a	d
d	a	b	c

2a)

a	**c**	d	b
d	b	a	c
b	a	c	d
c	d	b	a

2b)

a	**d**	b	c
c	b	a	d
d	a	c	b
b	c	d	a

3a)

a	**c**	d	b
d	b	c	a
c	a	b	d
b	d	a	c

3b)

a	**d**	c	b
c	b	a	d
d	c	b	a
b	a	d	c

4a)

a	**c**	b	d
d	b	a	c
b	d	c	a
c	a	d	b

4b)

a	**d**	b	c
c	b	d	a
b	a	c	d
d	c	a	b

Though these boards do not initially seem familiar, we see that under the following permutations, these boards are, in fact, the exact representatives of equivalence class [B]:

1a) : $(bdc) \rightarrow$ IIc; 1b) : $(bd) \rightarrow$ IIIb; 2a) : $(bdc) \rightarrow$ IIIC; 2b) : $(bd) \rightarrow$ IIB;
3a) $(bdc) \rightarrow$ IB; 3b) : $(bd) \rightarrow$ IC; 4a) : $(bdc) \rightarrow$ IVB; 4b) : $(bd) \rightarrow$ IVC.

Then we have 8 board structures, each representing 24 permutations; then we see that there are exactly 192 possible boards with exactly three distinct digits along each main diagonal. As this is the exact size of [B], we recognize the structure inherent to the equivalence class.

We now see the reason for the two equivalence classes: no permutation $\in S_4$ can change the number of digits along a diagonal, and any of the group action transformations must change an even number of digits when counting along the diagonals. Further, we now see a distinctive way to classify boards: that of counting the distinct digits along each main diagonal. ◊

Conclusion

A variation on Latin squares, Sudoku as a puzzle was invented in the late 1970s [5]. As with many playful diversions, though its origins were not mathematical, sudoku boards as mathematical objects hold great opportunities for study. Indeed, it is surprising how much can be found in a sudoku board, and the many ways it can be studied. Certainly the puzzle maker(s) who created "Number Place" had no intention that a board be encoded as a graph, or that chromatic polynomials may indicate the number of valid solutions for a given puzzle.

Regarding the chromatic polynomial as relates to sudoku graphs, the partial chromatic polynomial is an object deserving further study. While it has been shown that this polynomial is monic of known degree, our investigation has not shed light on its further properties, those shared with or different from the classic chromatic polynomial. For instance, is every power of λ present (up to the degree)? Must the signs of the terms alternate? What is the meaning, if any, of the constant term? Why does this polynomial have a constant term at all? These questions, beyond our scope, pique the curiosity.

We have used 4x4 boards as our object of study; in the world of sudoku, these are small boards, and in the mathematical study of sudoku, 4x4 is the trivial case. Given an opportunity, this author intends to study sudoku further, including 9x9 boards; in the meantime, we shall not forget that this fascinating mathematical object is also an enjoyable puzzle.

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